

The University of Chicago

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A DISSERTATION SUBMITTED TO THE FACULTY OF THE
DIVISION OF THE PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF METEOROLOGY
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THE UNIVERSITY OF LEIDEN

A QUASI-FLUID MECHANICS
SYSTEM OF HYDRODYNAMICAL
EQUATIONS

BY
J. VAN VLIET
AND
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By Victor P. Starr

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ABSTRACT

In this paper a system of hydrodynamical equations is developed which utilizes as independent variables two space coordinates and a material coordinate together with time. The space coordinates retain the same significance as the corresponding space coordinates in the Eulerian system of equations, while the material coordinate has properties identical with one of the material coordinates in the Lagrangian system. The system here presented is therefore one of the possible hybrid combinations of the two classic systems. A Cartesian reference framework is used throughout, although other reference coordinates could be used. The fundamental principles are formulated, including various forms of the equation of continuity and the equations of motion.

The main purpose of the system here described is to render certain geophysical problems more directly tractable for analysis and solution.

1. Introduction. In the study of the dynamics of atmospheric and oceanic motions it has been customary for investigators to take over and make use of the systems of equations set forth in the field of classical hydrodynamics, making only such minor changes in them as are obviously needed to adapt them for the purpose at hand. The two systems thus appropriated are the Eulerian ((4) pp. 2-12) which seeks to describe the properties of the motion at fixed points in space as time progresses, and the Lagrangian ((4) pp. 12-15) which seeks to state the locations of individual particles as time elapses (see also (2)). Since the motions of the atmosphere or of the oceans must obey the laws of fluid mechanics, it follows that either of these systems can be used to describe these motions. However, in the writer's opinion, it does not follow that in certain problems either one of these two approaches is the most convenient or natural one to use, because it is true that additional ones having certain advantages can be formulated.

One of the fundamental difficulties encountered in the analytical study of motions in the atmosphere and in the oceans is that the media are not homogeneous in the hydrodynamical sense of the term (i.e., the density is not a unique function of the pressure). Thus in the ocean we ordinarily find a stratification present so that isopycnic surfaces* assume a quasi-horizontal distribution, while in the atmosphere the surfaces of constant potential temperature are so oriented. Only a relatively small amount of attention has been given to problems involving such media in

* More precisely, the lack of homogeneity in the oceans is due to a combination of factors the most important being varying temperature and salinity. The matter is further complicated by the fact that sea water is slightly compressible. According to Sverdrup (11) the quantity which represents very nearly the analogue of the potential temperature in a moisture-free atmosphere is, in the case of the oceans, the density of the water at atmospheric pressure usually expressed as an anomaly and indicated by the symbol σ_t .

classical hydrodynamics, and it thus becomes necessary for geophysicists to devise appropriate means to treat the hydrodynamics of heterogeneous fluids.†

In the following pages a composite system of equations utilizing features of both the Lagrangian and the Eulerian system is developed. Its prime purpose is to render the treatment of heterogeneous fluid motion more tractable in connection with geophysical problems where the stratification present is essentially, but rarely if ever exactly, horizontal. In principle this approach has already been used for this purpose with considerable success by Dr. C.-G. Rossby and several others‡ in various investigations in meteorology and oceanography. In addition the writer has found this composite system quite useful for the study of certain problems involving the flow of homogeneous fluids. It is felt, however, that a systematic development of the method *ab initio* will be of some value in clarifying the concepts involved.

2. Basic concepts. Let us picture the fluid under consideration as moving in space relative to a Cartesian coordinate system fixed with reference to the earth and rotating with the angular velocity of the earth. We will assume that the x and the y axes are horizontal and point toward the east and north respectively, while the z axis is vertical and is counted positive upward. For some purposes the orientation of the horizontal axes is immaterial, but we shall find it convenient not to alter the axes of reference otherwise.

At some particular instant of time we next picture the space as being occupied by a family of nonintersecting surfaces which do not depart very much from the horizontal, defined purely arbitrarily in geometric fashion. Let us now assign identifying numbers to all

† Thus, for example, V. Bjerknes (2) developed the circulation theorem which bears his name.

‡ See for example references (5), (6), (7), (8) and (9).

members of this family of surfaces in consecutive fashion but otherwise arbitrarily. Thus the identifying parameter will be denoted by c so that associated with a particular value of c there is one and only one surface. The quantity c will moreover be taken as a *continuous* variable.

At the instant which we are considering, a given surface marks the location of a certain set of fluid particles. We can thus say that we have assigned an identifying number c to this set of particles. As time elapses we shall next consider that *the particles retain their identifying numbers* so that the surfaces along which c is constant in general become deformed in a way determined by the motion of the fluid. In the large scale motions of the atmosphere and of the oceans these surfaces would not in general be deformed very much from the horizontal, however, except over relatively long periods of time.

The fundamental question now is how can we determine the height z_1 of any given surface c_1 at a given geographical location x_1, y_1 and at some desired time t_1 . In order to answer this question it is necessary to know what function the *dependent variable* z is of the *independent variables* x, y, c and t . The form of this function in a particular problem must be determined by a solution of the hydrodynamical equations which we shall now discuss.

Although we have considered z as being the fundamental dependent variable there may be other quantities whose distributions in terms of x, y, c and t might be desired, such as pressure, temperature, humidity or any other continuously varying property. If we denote by α any such property, then

$$d\alpha = \frac{\partial\alpha}{\partial x} dx + \frac{\partial\alpha}{\partial y} dy + \frac{\partial\alpha}{\partial c} dc + \frac{\partial\alpha}{\partial t} dt. \quad (1)$$

If we wish to restrict the change $d\alpha$ to mean the change in α for a moving individual particle, it is clear that dc must be zero, since c is defined as being constant during the life history of a given parcel. Henceforth this meaning of $d\alpha$ will be assumed. Dividing by dt we have (see (6))

$$\frac{d\alpha}{dt} = u \frac{\partial\alpha}{\partial x} + v \frac{\partial\alpha}{\partial y} + \frac{\partial\alpha}{\partial t}; \quad \left(\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v \right). \quad (2)$$

Here $d\alpha/dt$ is the individual time rate of change of α for a particle, $\partial\alpha/\partial t$ is the time rate of change of α at a point (which may move vertically) for which x, y and c , are constant, $\partial\alpha/\partial x$ is the rate of change of α along x taken in such a manner that y, c and t are constant, $\partial\alpha/\partial y$ is a similar quantity, and u and v are the horizontal components of the velocity of an individual particle along the x and y directions respectively.

It is to be noted that since the vertical coordinate z is now a dependent variable which may be expressed as a function of x, y, c and t , it follows that the vertical velocity w of a particle may be expressed as follows:

$$w \equiv \frac{dz}{dt} = \frac{\partial z}{\partial t} + u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial y}. \quad (3)$$

3. Equation of continuity. Consider an element of volume with vertical sides and *horizontal* dimensions δx and δy and bounded by the surfaces $c = c_1$, and $c = c_1 + \delta c$. The geometric vertical thickness δz is then equal to $(\partial z/\partial c)\delta c$. The area of the horizontal projection of the element is $\delta x \delta y$, and the volume is then given by

$$\text{Volume} = \frac{\partial z}{\partial c} \delta x \delta y \delta c.$$

If ρ is the density, the mass is given by

$$\text{Mass} = \rho \frac{\partial z}{\partial c} \delta x \delta y \delta c.$$

The *individual* rate of change of this mass must be zero so that at the instant considered (see next section)

$$\frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \delta x \delta y \delta c \right) = 0 \quad (4)$$

and since $d/dt(\delta c) = 0$,

$$\begin{aligned} \delta x \delta y \frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \right) &= \\ &= - \rho \frac{\partial z}{\partial c} \left[\delta y \frac{d}{dt} (\delta x) + \delta x \frac{d}{dt} (\delta y) \right]. \end{aligned} \quad (5)$$

Now δx and δy are changing because of velocity gradients *along material surfaces of constant c* so that

$$\left. \begin{aligned} \frac{d}{dt} (\delta x) &= \frac{\partial u}{\partial x} \delta x, \\ \frac{d}{dt} (\delta y) &= \frac{\partial v}{\partial y} \delta y. \end{aligned} \right\} \quad (6)$$

The equation thus becomes

$$\frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \right) = - \rho \frac{\partial z}{\partial c} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (7)$$

which is one form of the desired relationship.

In the above (4) could be written simply as

$$\frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} A \delta c \right) = 0 \quad (8)$$

where A is the area of the horizontal projection of the element, whatever its shape so that

$$A \frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \right) = - \rho \frac{\partial z}{\partial c} \frac{dA}{dt}. \quad (9)$$

The quantity dA/dt may now be interpreted as the area multiplied by the divergence of the horizontal velocity components (along a constant c surface) so that equation (7) is again obtained. It should be noted that no assumption as to compressibility or homogeneity has been made, nor has any special property been assigned to the fluid so that (7) is a perfectly general relationship.

By expanding the total derivative in (7) according to (2) and collecting terms, an alternate general form is obtained, namely,

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial c} \right) + \frac{\partial}{\partial x} \left(\rho u \frac{\partial z}{\partial c} \right) + \frac{\partial}{\partial y} \left(\rho v \frac{\partial z}{\partial c} \right) = 0. \quad (10)$$

It should be kept in mind that the independent variable c may be chosen in an arbitrary fashion as in the Lagrangian system of hydrodynamical equations. In special problems it is permissible to choose it in special ways. Thus in a stratified atmosphere moving *adiabatically*, it may be identified with the conservative property the potential temperature θ , so that

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial \theta} \right) + \frac{\partial}{\partial x} \left(\rho u \frac{\partial z}{\partial \theta} \right) + \frac{\partial}{\partial y} \left(\rho v \frac{\partial z}{\partial \theta} \right) = 0, \quad (11)$$

or

$$\frac{d}{dt} \left(\rho \frac{\partial z}{\partial \theta} \right) = - \rho \frac{\partial z}{\partial \theta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (12)$$

Again, if the fluid is stratified and $d\rho/dt = 0$, the density is a conservative property and surfaces of constant c may be thought of as isopycnic sheets so that c may be identified with ρ . In this case we have

$$\frac{\partial^2 z}{\partial t \partial \rho} + \frac{\partial}{\partial x} \left(u \frac{\partial z}{\partial \rho} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial z}{\partial \rho} \right) = 0, \quad (13)$$

or

$$\frac{d}{dt} \left(\frac{\partial z}{\partial \rho} \right) = - \frac{\partial z}{\partial \rho} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (14)$$

In the case just mentioned if c were not chosen so as to be constant along the isopycnic sheets, an equation of the form of (13) would still be true.* This can be seen from equation (7) which now takes the form

$$\frac{d}{dt} \left(\frac{\partial z}{\partial c} \right) = - \frac{\partial z}{\partial c} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (15)$$

This on expansion etc., gives

$$\frac{\partial^2 z}{\partial t \partial c} + \frac{\partial}{\partial x} \left(u \frac{\partial z}{\partial c} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial z}{\partial c} \right) = 0. \quad (16)$$

Equation (13) (or equation (16)) admits a further modification of general interest if the vertical ve-

locity w is introduced explicitly. From (3)

$$\frac{\partial z}{\partial t} = w - u \frac{\partial z}{\partial x} - v \frac{\partial z}{\partial y}.$$

Differentiating with respect to ρ partially

$$\frac{\partial^2 z}{\partial t \partial \rho} = \frac{\partial w}{\partial \rho} - u \frac{\partial^2 z}{\partial x \partial \rho} - v \frac{\partial^2 z}{\partial y \partial \rho} - \frac{\partial u}{\partial \rho} \frac{\partial z}{\partial x} - \frac{\partial v}{\partial \rho} \frac{\partial z}{\partial y} \quad (17)$$

Rewriting (13) we have

$$\frac{\partial^2 z}{\partial t \partial \rho} + u \frac{\partial^2 z}{\partial x \partial \rho} + v \frac{\partial^2 z}{\partial y \partial \rho} + \frac{\partial u}{\partial x} \frac{\partial z}{\partial \rho} + \frac{\partial v}{\partial y} \frac{\partial z}{\partial \rho} = 0.$$

Combining this with (17) so as to eliminate the time derivatives,

$$\frac{\partial w}{\partial \rho} + \frac{\partial(u, z)}{\partial(x, \rho)} + \frac{\partial(v, z)}{\partial(y, \rho)} = 0. \quad (18)$$

More generally, if (3) be first multiplied by ρ and then differentiated partially with respect to c , not making any special assumptions, the result when combined with (10) gives

$$\frac{\partial}{\partial c} (\rho w) + \frac{\partial(\rho, z)}{\partial(t, c)} + \frac{\partial(\rho u, z)}{\partial(x, c)} + \frac{\partial(\rho v, z)}{\partial(y, c)} = 0. \quad (19)$$

If the medium is in hydrostatic equilibrium and p is pressure and g is the acceleration of gravity,† the hydrostatic equation may be transformed as follows:

$$0 = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} = -g - \frac{1}{\rho} \frac{\partial p}{\partial c} \frac{\partial c}{\partial z},$$

since for a given vertical z is a function of c alone. Also we may now write

$$\rho \frac{\partial z}{\partial c} = - \frac{1}{g} \frac{\partial p}{\partial c}. \quad (20)$$

Substitution from (20) into (10) now gives (assuming g to be constant)

$$\frac{\partial^2 p}{\partial t \partial c} + \frac{\partial}{\partial x} \left(u \frac{\partial p}{\partial c} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial p}{\partial c} \right) = 0, \quad (21)$$

or

$$\frac{d}{dt} \left(\frac{\partial p}{\partial c} \right) = - \frac{\partial p}{\partial c} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (22)$$

If these are written in slightly different form we have‡

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\partial p}{\partial c} \right) + \frac{\partial p}{\partial c} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \\ & + u \frac{\partial}{\partial x} \left(\frac{\partial p}{\partial c} \right) + v \frac{\partial}{\partial y} \left(\frac{\partial p}{\partial c} \right) = 0. \end{aligned} \quad (23)$$

* Although it often is convenient to identify c with some conservative physical property or a unique function of such a property, this choice is not essential. In the case of homogeneous fluids, for example, this cannot generally be done and c must be defined geometrically.

† We shall regard g as constant, although the subsequent material could be modified and written in terms of geopotential, thus allowing for variations in g .

‡ Equation (23) may be applied to the theory of "isentropic weight charts" used in meteorology, in order to illustrate its

It is possible to write the equation of continuity in various other ways in addition to the above forms by ascribing thermodynamic properties to the fluid considered. This has been done to some extent in introducing the potential temperature, but this procedure may be extended, for instance by the use of the equation of state.

4. **An alternate method for the derivation of the continuity equation.** The previous derivation of the continuity equation is open to certain objections. Thus the area of the horizontal projection of the element previously taken cannot be considered as a rectangle as time passes, although it could be shown that it is sufficient to consider its behavior at the instant when it is a rectangle. For this and other reasons it is desirable to obtain the final equation by considering a finite portion of the fluid and using integral relationships. In order to accomplish this it is necessary to establish the concept of a mapping process whereby the conditions in the actual space (which we call the x, y, z space) are mapped in a fictitious space defined by a Cartesian coordinate system having coordinates x, y and c . In this fictitious space (henceforth called the x, y, c space) the horizontal coordinates x and y remain the same as in the actual space, but the vertical coordinate is replaced by the quantity c counted on a linear scale. It is obvious that the surfaces of constant c will map into planes parallel to the x, y plane in the x, y, c space. Furthermore since all the particles retain their characteristic values of c , all motions in the x, y, c space will be parallel to the x, y plane. Because the mapping process in general either stretches the surfaces of constant c apart or shrinks material layers vertically, we may consider that the actual density ρ is mapped into a fictitious density σ related to the actual density by the following expression

$$\sigma = \rho \frac{\partial z}{\partial c}. \quad (24)$$

Let us now consider a volume not necessarily small in the x, y, z space located entirely within the fluid. The mass M contained in this volume is given by

$$M = \iiint \rho \delta x \delta y \delta z.$$

In the mapped (x, y, c) space the expression for M significance. Assuming the motions to be adiabatic, $\partial p / \partial c$ can be replaced *approximately* by the ratio of Δp to some constant potential temperature difference $\Delta \theta$. Since the quantity $1/\Delta \theta$ will then appear as a constant linear factor in all terms, (23) reduces to

$$\frac{\partial}{\partial t} (\Delta p) = -\Delta p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - u \frac{\partial}{\partial x} (\Delta p) - v \frac{\partial}{\partial y} (\Delta p).$$

It is apparent that a given rate of change of Δp at a particular vertical may be the result either of convergence of the horizontal velocity as defined in our system or of "advection," or of a combination of both these processes.

becomes

$$M = \iiint \rho \frac{\partial z}{\partial c} \delta x \delta y \delta c, \quad (25)$$

with the appropriate limits of integration which we will assume to remain fixed with respect to time. Thus if we differentiate (25) with respect to time partially, so that

$$\frac{\partial M}{\partial t} = \iiint \frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial c} \right) \delta x \delta y \delta c, \quad (26)$$

the result is the rate of change of mass within a fixed volume in the x, y, c space.

We may obtain another expression for the left hand side of (26) by integrating the lateral inflow of mass into the fixed volume in the x, y, c space. Thus

$$\frac{\partial M}{\partial t} = \iint \rho \frac{\partial z}{\partial c} V \cdot \delta s \quad (27)$$

where V is the vector velocity in the x, y, c space and δs is a vector element of the surface area taken along the inward normal. We may now make use of the divergence theorem and state that

$$\begin{aligned} \frac{\partial M}{\partial t} &= \iiint \rho \frac{\partial z}{\partial c} V \cdot \delta s = \\ &= - \iiint \operatorname{div} \left(\rho \frac{\partial z}{\partial c} V \right) \delta x \delta y \delta c. \end{aligned} \quad (28)$$

Combining (28) with (26),

$$\begin{aligned} \iiint \left[\frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial c} \right) + \right. \\ \left. + \operatorname{div} \left(\rho \frac{\partial z}{\partial c} V \right) \right] \delta x \delta y \delta c = 0. \end{aligned} \quad (29)$$

Since this must be true no matter what the limits of integration may be, the only possibility is that the integrand is identically zero. Further, since the velocity in the x, y, c space is parallel to the x, y plane with components which are the same as the horizontal velocity components in the x, y, z space we again obtain equation (10), namely

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial c} \right) + \frac{\partial}{\partial x} \left(\rho u \frac{\partial z}{\partial c} \right) + \frac{\partial}{\partial y} \left(\rho v \frac{\partial z}{\partial c} \right) = 0. \quad (10)$$

5. **Use of the initial coordinates of particles.** Thus far no mention has been made of utilizing the initial coordinates of particles, at time $t = 0$, in the x, y, z space for purposes of analysis. These quantities, indicated by the symbols x_0, y_0 and z_0 , may serve as conservative properties of particles as in the Lagrangian system of equations. The quantity z_0 can be used as the independent variable c in certain problems where this meaning of c is convenient. The quantities

x_0 and y_0 , on the other hand, we must regard as dependent variables whose distributions are to be specified in terms of x , y , c and t . We will now see what form the equation of continuity takes if we introduce the variables x_0 , y_0 and z_0 .

Let us write (9) in the form

$$\frac{1}{\rho} \frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \right) = - \frac{1}{A} \frac{dA}{dt}, \quad (9)$$

where A is the area of the horizontal projection of an individual element in a sheet of constant c . If the element has a rectangular projection at time $t = 0$, of dimensions δx_0 and δy_0 , at a later time its projection will be represented by a parallelogram, to the first order of approximation. The length of the projections of the sides of this parallelogram on the x and y axes will be

$$\begin{aligned} \frac{\partial x}{\partial x_0} \delta x_0, & \quad \frac{\partial y}{\partial x_0} \delta x_0 \\ \frac{\partial x}{\partial y_0} \delta y_0, & \quad \frac{\partial y}{\partial y_0} \delta y_0 \end{aligned}$$

in terms of partial derivatives with respect to x_0 and y_0 . Since the area of a parallelogram is the magnitude of the vector cross product of its sides taken as vectors, it follows that the new area is given by the above four quantities in the form of a determinant in the order given. Thus

$$A = \left(\frac{\partial x}{\partial x_0} \frac{\partial y}{\partial y_0} - \frac{\partial y}{\partial x_0} \frac{\partial x}{\partial y_0} \right) \delta x_0 \delta y_0. \quad (30)$$

Since we wish to regard x_0 and y_0 as dependent variables, we may make use of the property of Jacobians which states that

$$\frac{\partial(x, y)}{\partial(x_0, y_0)} = \frac{1}{\frac{\partial(x_0, y_0)}{\partial(x, y)}}.$$

We then have that

$$A = \frac{\delta x_0 \delta y_0}{\frac{\partial(x_0, y_0)}{\partial(x, y)}} \quad (31)$$

Placing the definition of A contained in (31) into (9) we get

$$\frac{1}{\rho} \frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \right) = \frac{1}{\frac{\partial(x_0, y_0)}{\partial(x, y)}} \frac{d}{dt} \left[\frac{\partial(x_0, y_0)}{\partial(x, y)} \right], \quad (32)$$

since $\delta x_0 \delta y_0$ is an individual constant. Individual integration of (32) yields

$$\rho \frac{\partial z}{\partial c} = K \left(\frac{\partial x_0}{\partial x} \frac{\partial y_0}{\partial y} - \frac{\partial y_0}{\partial x} \frac{\partial x_0}{\partial y} \right), \quad (33)$$

in which K is a constant of integration. Since (33) must be valid also for the initial instant when $x = x_0$, $y = y_0$, $z = z_0$ and ρ has its initial value ρ_0 , it follows that $K = \rho_0 \partial z_0 / \partial c$, and the final form of the continuity equation is

$$\rho \frac{\partial z}{\partial c} = \rho_0 \frac{\partial z_0}{\partial c} \left(\frac{\partial x_0}{\partial x} \frac{\partial y_0}{\partial y} - \frac{\partial y_0}{\partial x} \frac{\partial x_0}{\partial y} \right). \quad (34)$$

This equation is still in general form, since no special properties were assumed for the motion or for the characteristics of the fluid. Also, it should be noted that time does not enter the relation explicitly; the equation only relates two possible configurations of the fluid.

In special problems various simplifications of (34) may occur. Thus if, as sometimes is desired, a system of flow is studied in which material chains of particles stay parallel to one of the horizontal axes, let us say the x axis, then $\partial y_0 / \partial x$ vanishes. If, further, material particles maintain their relative distances along the x direction, $\partial x_0 / \partial x = 1$ and our equation becomes

$$\rho \frac{\partial z}{\partial c} = \rho_0 \frac{\partial z_0}{\partial c} \frac{\partial y_0}{\partial y}. \quad (35)$$

As an extreme let us suppose further that the fluid moves without individual changes in density (but may be heterogeneous), and that we select c to be equal to z_0 . It then follows that

$$\frac{\partial z}{\partial z_0} = \frac{\partial y_0}{\partial y}. \quad (36)$$

If the fluid is in hydrostatic equilibrium the combining of (20) with (34) gives (assuming g to be constant)

$$\frac{\partial p}{\partial c} = \frac{\partial p_0}{\partial c} \left(\frac{\partial x_0}{\partial x} \frac{\partial y_0}{\partial y} - \frac{\partial y_0}{\partial x} \frac{\partial x_0}{\partial y} \right), \quad (37)$$

in which p_0 is the initial pressure.

6. Integral forms of the continuity equation. In some problems it is permissible to assume that the horizontal components of the motion of a fluid are independent of the variable c . The validity of such an assumption must however be established by an appeal to the *dynamics* of the flow system. At present we shall see what forms the continuity equation takes under such simplification. Equation (7) can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial c} \right) + u \frac{\partial}{\partial x} \left(\rho \frac{\partial z}{\partial c} \right) + \\ + v \frac{\partial}{\partial y} \left(\rho \frac{\partial z}{\partial c} \right) = - \rho \frac{\partial z}{\partial c} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \end{aligned}$$

If we represent the mass per unit vertical column

between c_1 and c_2 by M , so that

$$M = \int_{c_1}^{c_2} \rho \frac{\partial z}{\partial c} \delta c = \int_{z_1}^{z_2} \rho \delta z,$$

it is apparent that (7) can be integrated so as to give

$$\frac{\partial M}{\partial t} + u \frac{\partial M}{\partial x} + v \frac{\partial M}{\partial y} = -M \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (38)$$

or

$$\frac{dM}{dt} = -M \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (39)$$

provided only that u and v are not functions of c within the *individual layer* of fluid bounded by $c = c_1$ and $c = c_2$. These bounding surfaces may be the bottom and the free surface in the case of a liquid. Since the motion is the same at all levels the divergence at the right may be evaluated along any surface which is not vertical but which is within the given fluid layer. If the fluid is in hydrostatic equilibrium and g is assumed constant the mass M is proportional to the pressure difference Δp along a vertical between c_1 and c_2 and (39) becomes

$$\frac{d}{dt}(\Delta p) = -\Delta p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (40)$$

If $d\rho/dt = 0$, the density can be omitted from (7) (although the fluid may still have space variations of density). We now can introduce the depth D of an individual fluid layer between $c = c_1$ and $c = c_2$ so that

$$D = \int_{c_1}^{c_2} \frac{\partial z}{\partial c} \delta c.$$

Again through integration we obtain

$$\frac{dD}{dt} = -D \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (41)$$

or the forms

$$\frac{\partial D}{\partial t} + u \frac{\partial D}{\partial x} + v \frac{\partial D}{\partial y} = -D \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (42)$$

and

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial x}(uD) + \frac{\partial}{\partial y}(vD) = 0. \quad (43)$$

A similar treatment can be applied to equation (34) giving

$$M = M_0 \left(\frac{\partial x_0}{\partial x} \frac{\partial y_0}{\partial y} - \frac{\partial y_0}{\partial x} \frac{\partial x_0}{\partial y} \right) \quad (44)$$

which for incompressible motion becomes*

$$D = D_0 \left(\frac{\partial x_0}{\partial x} \frac{\partial y_0}{\partial y} - \frac{\partial y_0}{\partial x} \frac{\partial x_0}{\partial y} \right). \quad (45)$$

* This equation has been used by Rossby (8) in a study of readjustments of mass distribution in the ocean.

Equation (36) becomes the simple relation

$$D = D_0 \frac{\partial y_0}{\partial y}, \quad (46)$$

which could have been derived by simpler considerations.

Forms (41), (42), (43), (45) and (46) are usually applied to fluids of constant density throughout, but *kinematic* considerations alone do not impose so stringent an assumption.

7. Stream function for material surfaces. In the Eulerian system of hydrodynamics it is possible to define a function ψ_1 (a stream function) such that for two-dimensional motion, for example in the x, y plane

$$u = -\frac{\partial \psi_1}{\partial y}, \quad v = +\frac{\partial \psi_1}{\partial x},$$

provided that the divergence of the horizontal velocity field is identically zero at all values of x, y and t considered. Our present purpose will be to investigate whether an analogue of this concept of a stream function exists also in the system that has been proposed in the preceding discussions.

For this purpose it will again be convenient to make use of the concept of the x, y, c space in which the coordinates are orthogonal and in which, by virtue of the manner in which c is defined, all velocities are parallel to the x, y plane. Let us focus attention on a horizontal sheet of the medium in this space, of a constant "thickness" δc , bounded above and below by surfaces of constant value of c .

At some given instant let us next consider volume transport in the x, y, c space within this sheet across some arbitrary line joining the points O and P and fixed in position with respect to the coordinates (see Figure 1). Taking this transport as positive from right to left, its magnitude will be given by the line

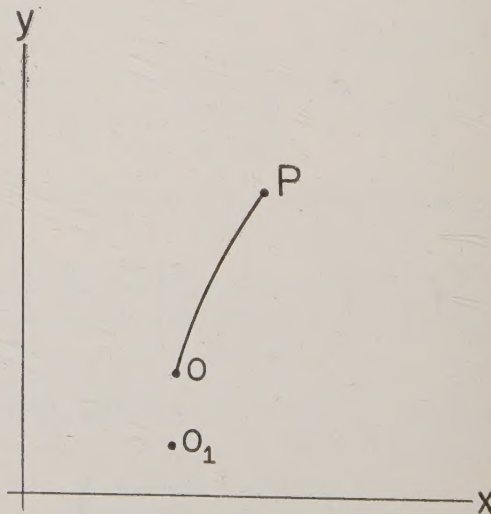


FIGURE 1.

integral of the velocity normal to the line multiplied by δc . If we now designate the transport by $\psi \delta c$ we have

$$\psi \delta c = \delta c \int_0^P (v \delta x - u \delta y). \quad (47)$$

Since δc is a constant it may be factored from the equation. If ψ as defined by (47) is to be uniquely specified irrespective of what particular line is used to join 0 and P , it follows that $v \delta x - u \delta y$ must be an exact differential. The condition for its exactness is that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (48)$$

Thus, considering various positions for point P in the x, y plane we see that with reference to point 0, it is possible to completely map out a unique field of ψ values for a given state of motion. Moreover, the locus of point P prescribed by a given constant value of ψ constitutes a line across which no transport takes place in any portion, and which therefore has no velocity components normal to it. Such a line must therefore constitute a streamline. In the actual space this line will have the same horizontal direction as the horizontal velocity.

Since the point 0 has been chosen arbitrarily, a different choice for this point such as 0_1 would lead to a field of ψ differing by a constant amount from the previous one; this amount, $\Delta\psi$, being given by

$$\Delta\psi = \int_{0_1}^0 (v \delta x - u \delta y). \quad (49)$$

It is thus seen that if the point of reference is not specified, the stream function will be unspecified to the extent of an unknown additive constant. However, lack of knowledge of this additive constant introduces no arbitrariness concerning the pattern of the streamlines.

We may next write the relation that

$$\delta\psi = v \delta x - u \delta y. \quad (50)$$

If a line element is so chosen that it is parallel to one or the other horizontal axis we obtain the relations that

$$u = -\frac{\partial\psi}{\partial y}, \quad v = +\frac{\partial\psi}{\partial x}. \quad (51)$$

The necessary condition for the existence of a stream function of the above type,* given by (48), is capable of further physical interpretation if reference is made to the equation of continuity appropriate for the kind of fluid considered. Thus comparison of (48)

* The function ψ discussed here is not to be confused with the quantity $c_p T + gz$ used in meteorology to obtain the gradient wind in an isentropic surface. The quantity ψ as used here is defined purely kinematically and is related only to the continuity equation.

and (7) leads to the requirement that the quantity $\rho \partial z / \partial c$ be constant for individual particles, although this quantity need not have the same value for all particles in the material sheet considered. Other similar observations can be made by reference to equations (14), (15) and (22).

The function ψ discussed above was derived through the consideration of the line integral of the normal velocity across an arbitrary line segment connecting two points on a surface of constant c . This process corresponds directly to the procedure used to establish the concept of a stream function in the Eulerian system. For this reason the function ψ was alluded to simply as a stream function even though its meaning is somewhat more general than that of the Eulerian form, in that the surface of constant c need not be a plane surface. However, in our system (as well as in the Eulerian) it is possible to define a function similar to ψ in that its isolines are parallel to the horizontal velocity vectors as far as azimuth is concerned, but with different other properties, by considering the line integral not of the normal velocity itself but of the normal velocity weighted by certain scalar factors. This function will be termed a "flow function," and denoted by the symbol η . By using the same scheme as in the demonstration of the stream function ψ we now write

$$\eta \delta c = \delta c \int_0^P \left(\rho v \frac{\partial z}{\partial c} \delta x - \rho u \frac{\partial z}{\partial c} \delta y \right). \quad (52)$$

The condition for exactness is now:

$$\frac{\partial}{\partial x} \left(\rho u \frac{\partial z}{\partial c} \right) + \frac{\partial}{\partial y} \left(\rho v \frac{\partial z}{\partial c} \right) = 0. \quad (53)$$

If this is true we may state that

$$\delta\eta = \rho v \frac{\partial z}{\partial c} \delta x - \rho u \frac{\partial z}{\partial c} \delta y, \quad (54)$$

from which we get that

$$\rho u \frac{\partial z}{\partial c} = -\frac{\partial\eta}{\partial y}, \quad \rho v \frac{\partial z}{\partial c} = +\frac{\partial\eta}{\partial x}. \quad (55)$$

The modifications of the equations (52)–(55) for fluids with special properties may be readily obtained. For instance in the case that the density is not a function of x and y , (52) may be divided by ρ and the factors $1/\rho$ be absorbed in η .

One fact to be remarked is that as in the case of the stream function the condition for integrability (53) when imposed on the continuity equation (10) leads to certain requirements, namely in this case that

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial z}{\partial c} \right) = 0. \quad (56)$$

This has the significance that in the x, y, c space at a fixed point, $\rho \partial z / \partial c$ must have the same value as time progresses, or that in the x, y, z space the above quantity cannot change value with time at a point having fixed value of x, y and c , however the position of the point may change along the z coordinate. In the case that the density is constant along the c surface the above remarks apply to the geometric thickness, δz , of an elementary material layer.

The preceding discussions of the stream function and flow function show that at any particular instant the distribution of these scalar quantities on a surface of constant c enables us to trace out lines which have the same (horizontal) direction as the horizontal velocity vectors, provided certain conditions are fulfilled. There is, however, no stipulation which would preclude the special case from arising in which the surface of constant c happens to coincide, momentarily at least, with a surface of constant physical or geometric property of the fluid, whether this property be conservative or not. It is thus legitimate to speak of a stream or flow function distribution on an isothermal or isobaric surface, for example.

An alternative way of justifying this concept is as follows: In our scheme of independent variables it is necessary to assign values of c to all the individual particles at some instant of time, and thereafter those values are retained by the particles during their motion. There is no reason, therefore, why the values of c cannot be assigned at a given instant in such a manner that a given surface of constant c coincides with a surface of constant pressure or other property.

8. The equations of motion. In order to formulate the dynamical equations of motion appropriate for our scheme of independent variables, it is convenient to simply change variables in the corresponding Eulerian form of these equations. These latter are

$$\frac{du}{dt} = fv - jw - \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (57)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y}, \quad (58)$$

$$\frac{dw}{dt} = -g + ju - \frac{1}{\rho} \frac{\partial p}{\partial z}. \quad (59)$$

Here $f = 2\Omega \sin \phi$ and $j = 2\Omega \cos \phi$ where Ω is the angular velocity of the earth's rotation and ϕ is latitude. We shall not attempt to incorporate frictional terms in the equations of motion. The significance of the quantities u, v, w and $du/dt, dv/dt, dw/dt$ is the same in our system, so that the terms containing these quantities will not be altered. It should however be noted that the expansion of the individual derivatives is to be carried out according to equation (2) in our system. The terms representing the pressure

gradient force components require modification so that *the same three quantities* will be represented by expressions containing derivatives with respect to our independent variables x, y and c . We have already mentioned that for any horizontal direction, for instance the x direction,

$$\left(\frac{\partial \alpha}{\partial z} \right)_x = \frac{\left(\frac{\partial \alpha}{\partial c} \right)_x}{\left(\frac{\partial z}{\partial c} \right)_x}, \quad (60)$$

where α is any continuous variable and the subscripts indicate that the differentiation is performed on the assumption that the designated variable is held constant. Likewise by simple geometric considerations it can be shown that

$$\left(\frac{\partial \alpha}{\partial x} \right)_z = \left(\frac{\partial \alpha}{\partial x} \right)_c - \left(\frac{\partial \alpha}{\partial z} \right)_x \left(\frac{\partial z}{\partial x} \right)_c. \quad (61)$$

Combining (60) and (61)

$$\left(\frac{\partial \alpha}{\partial x} \right)_z = \left(\frac{\partial \alpha}{\partial x} \right)_c - \left(\frac{\partial z}{\partial x} \right)_c \frac{\left(\frac{\partial \alpha}{\partial c} \right)_x}{\left(\frac{\partial z}{\partial c} \right)_x}. \quad (62)$$

Introducing relationships of this kind into the equations (57), (58) and (59) to transform the pressure gradient terms and dropping the subscripts, which are automatically understood in our system, we get

$$\rho \frac{\partial z}{\partial c} \frac{du}{dt} = \rho f v \frac{\partial z}{\partial c} - \rho j w \frac{\partial z}{\partial c} + \frac{\partial z}{\partial x} \frac{\partial p}{\partial c} - \frac{\partial p}{\partial x} \frac{\partial z}{\partial c}, \quad (63)$$

$$\rho \frac{\partial z}{\partial c} \frac{dv}{dt} = -\rho f u \frac{\partial z}{\partial c} + \frac{\partial z}{\partial y} \frac{\partial p}{\partial c} - \frac{\partial p}{\partial y} \frac{\partial z}{\partial c}, \quad (64)$$

$$\rho \frac{\partial z}{\partial c} \frac{dw}{dt} = -\rho g \frac{\partial z}{\partial c} + \rho j u \frac{\partial z}{\partial c} - \frac{\partial p}{\partial c}. \quad (65)$$

These are the general equations of motion in our system.

For many meteorological and oceanographic purposes it is permissible to neglect the terms involving j and also the vertical acceleration dw/dt . Under such circumstances the first two equations may be simplified by means of the third which then reduces to the hydrostatic equation. The result is

$$\frac{du}{dt} = +fv - \frac{1}{\rho} \frac{\partial p}{\partial x} - g \frac{\partial z}{\partial x}, \quad (66)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y} - g \frac{\partial z}{\partial y}, \quad (67)$$

$$0 = -\rho g \frac{\partial z}{\partial c} - \frac{\partial p}{\partial c}. \quad (68)$$

Here g is assumed to be constant.

If, further, the c surfaces can be so chosen as to make the density a function of the pressure alone along each such surface of constant c so that

$$\frac{1}{\rho} = \frac{\partial F}{\partial p}, \quad (69)$$

where $F = F(p, c)$, then the equations (66), (67) and (68) may be written as follows:*

$$\frac{du}{dt} = +fv - \frac{\partial}{\partial x}(F + gz), \quad (70)$$

$$\frac{dv}{dt} = -fu - \frac{\partial}{\partial y}(F + gz), \quad (71)$$

$$0 = -g \frac{\partial z}{\partial c} - \frac{1}{\rho} \frac{\partial p}{\partial c}. \quad (72)$$

If the fluid is homogeneous, $F = F(p)$ and (72) is equivalent to

$$0 = \frac{\partial}{\partial c}(F + gz). \quad (73)$$

9. A vorticity theorem and the circulation theorem.

Equations (70) and (71) may be rewritten in "hydrodynamical form" by the expansion of the individual derivatives so that

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = +fv \frac{\partial}{\partial x} - (F + gz), \quad (74)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -fu - \frac{\partial}{\partial y}(F + gz). \quad (75)$$

If proper terms are rearranged these two equations are equivalent to

$$\begin{aligned} \frac{\partial u}{\partial t} = & \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) v - \\ & - \frac{\partial}{\partial x} \left(\frac{u^2 + v^2}{2} + F + gz \right), \end{aligned} \quad (76)$$

* The form of the function F may be easily determined from the physical properties of the fluid considered. Thus if c is identified with the density (or is a given function of the density alone) in the case of a heterogeneous incompressible fluid, $F = p/\rho$. In the case of an atmosphere in which the c surfaces are identified with surfaces of constant potential temperature,

$$F = c_p \theta \left(\frac{p}{1000} \right)^{R/mc_p} = c_p T,$$

in which p is pressure measured in millibars, R/m is the gas constant, c_p is the specific heat of air at constant pressure, and T is the absolute temperature (see (5) and (6)).

In order that equations (70) and (71) apply at a given instant it is not essential that the c surfaces coincide with surfaces of a conservative property, provided that an equation of the form (69) holds at that instant of time. Thus they might momentarily coincide with isothermal surfaces in the atmosphere in which case (see (10)),

$$F = \frac{R}{m} T \ln p.$$

$$\begin{aligned} \frac{\partial v}{\partial t} = & - \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) u - \\ & - \frac{\partial}{\partial y} \left(\frac{u^2 + v^2}{2} + F + gz \right). \end{aligned} \quad (77)$$

Indicating the "vorticity" $\partial v/\partial x - \partial u/\partial y$ by ζ , we may obtain the following relationship by cross differentiating (76) and (77) and taking the difference of these derivatives so as to eliminate the terms containing F .

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + u \frac{\partial}{\partial x}(f + \zeta) + v \frac{\partial}{\partial y}(f + \zeta) = \\ = - (f + \zeta) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \end{aligned} \quad (78)$$

Since $\partial f/\partial t = 0$ we may write (78) in the form

$$\frac{d}{dt}(f + \zeta) = - (f + \zeta) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (79)$$

We shall call this equation the *vorticity theorem* although it is to be noted that the space variations of the velocity components are measured along material surfaces of constant c which need not be level surfaces. It is also to be remarked that under the assumptions made in deriving the equation no solenoid terms appear in it. If it is to apply over a finite length of time for an individual portion of fluid, it is necessary that the density be a function of pressure at every instant along each c surface, although the *form* of this function for each such surface may change as time passes. Thus this vorticity equation is still valid even if the fluid is thermodynamically active, provided that the above requirements are fulfilled, and may therefore be integrated with respect to time.

If (79) is combined with the continuity equation (7), the result, namely

$$\frac{1}{f + \zeta} \frac{d}{dt}(f + \zeta) = \frac{1}{\rho \frac{\partial z}{\partial c}} \frac{d}{dt} \left(\rho \frac{\partial z}{\partial c} \right), \quad (80)$$

is then capable of individual integration so that

$$\frac{f + \zeta}{\rho \frac{\partial z}{\partial c}} = \frac{f_0 + \zeta_0}{\rho_0 \frac{\partial z_0}{\partial c}}. \quad (81)$$

Here the subscript zero may denote the values of the several quantities involved at some specified time. Since hydrostatic equilibrium was assumed in its derivation, no generality is lost by combining (72) with (81) so as to give

$$\frac{f + \zeta}{\frac{\partial p}{\partial c}} = \frac{f_0 + \zeta_0}{\frac{\partial p_0}{\partial c}}. \quad (82)$$

Other modifications of (81) for fluids with special properties may easily be obtained.

From the equations of motion (63), (64) and (65) it is possible, as should be expected, to deduce the Bjerknes circulation theorem. For this purpose it is necessary to evaluate the individual rate of change of the circulation in a closed material chain of particles. It can be shown (see (3)) that this rate of change is equivalent to the line integral of the acceleration along the chain. Thus if C is the circulation

$$\frac{dC}{dt} = \oint \left(\frac{du}{dt} dx + \frac{dv}{dt} dy + \frac{dw}{dt} dz \right). \quad (83)$$

Since we are at liberty to choose the c surfaces at pleasure, let us choose them in such a way that the given circuit lies entirely within one such surface. Along the chain of particles we then have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \quad (84)$$

Here dx , dy and dz are space variations along the chain. If this is incorporated in (83),

$$\frac{dC}{dt} = \oint \left[\left(\frac{du}{dt} + \frac{\partial z}{\partial x} \frac{dw}{dt} \right) dx + \left(\frac{dv}{dt} + \frac{\partial z}{\partial y} \frac{dw}{dt} \right) dy \right]. \quad (85)$$

It will be convenient to deal with the absolute circulation and therefore to omit the Coriolis terms in substituting from the equations of motion (63), (64) and (65) for the accelerations in (85). The result is

$$\frac{dC}{dt} = - \oint \left[\frac{1}{\rho} \left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \right) + g \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right) \right], \quad (86)$$

or because of (84) and a similar relation for pressure,

$$\frac{dC}{dt} = - \oint \left(\frac{dp}{\rho} + g dz \right). \quad (87)$$

The term in (87) involving g vanishes whether or not g is taken as variable, because g is determined by the gradient of the geopotential and the geopotential must return to its initial value in going completely around the circuit.

We are thus led to the familiar form of the circulation theorem for absolute motion namely

$$\frac{dC}{dt} = - \oint \frac{dp}{\rho}. \quad (88)$$

This is of course capable of being transformed so as to be applicable to a rotating system as explained in

various textbooks on dynamic meteorology, and interpreted in various ways for special problems.

10. General remarks. The system of equations which has been presented represents the result of replacing the independent variable z in the Eulerian system by the quantity c which is one of the independent variables in the Lagrangian form of the hydrodynamical equations. It would also be possible to formulate a composite system by retaining only one coordinate of the Eulerian type and using two Lagrangian or material coordinates. The usefulness of such a system is however not very obvious.

Returning to the system presented, it is clear that it has been built upon a Cartesian coordinate system, although the vertical coordinate is regarded as a dependent variable. In similar fashion a system of equations could be formulated around a spherical polar framework, the radius being regarded as a dependent variable, while the latitude, the longitude and the variable c would be independent variables together with time. At least from a formal viewpoint such a system would possess a certain elegance for the study of large scale motions of the atmosphere but might be somewhat cumbersome as far as manipulation is concerned.

One question which may need additional comment is why must the quantity c be a conservative property. The main difficulty which would arise if c were not conservative relates to the expansion of an individual time derivative. Equation (2) would then have an additional term on the right involving the individual rate of change of c with time. This would introduce a quantity which generally speaking would become quite troublesome in the subsequent development of the subject. However, if we limit ourselves to topics which do not require the expansion of individual derivatives it is frequently possible to use a nonconservative quantity in the place of c . Thus for instance according to the geostrophic assumption frequently used in meteorology, horizontal accelerations are neglected and the resulting form of the equations of motion can then be easily transformed so as to apply to a surface of any constant property or even to a geometrically defined surface.*

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